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# On the perturbation bounds of g-inverses and oblique projections☆

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Let $A\{1\}$ and $\widehat{A}\{1\}$ be two sets of g-inverses of matrices $A$ and $\widehat{A} = A + E$, respectively. For any $A^- \in A\{1\}$, we deduce general formulas of g-inverse $\widehat{A}^- \in \widehat{A}\{1\}$, such that the distances between the two g-inverses or oblique projections are the smallest under appropriated norms, and obtain the corresponding distances. With these results, we derive perturbation bounds of the nearest perturbed g-inverses, oblique projections, and consistent linear equations under rank preserving condition $\text{rank}(\widehat{A}) = \text{rank}(A)$. Numerical examples are also provided to verify our analysis.



## 1. Introduction

Throughout this paper, we use the following notation. $\mathbf{C}^{m\times n}$ is the set of all $m \times n$ matrices with complex entries. For any matrix $A \in \mathbf{C}^{m\times n}$, $\text{rank}(A)$, $A^H$ denote the rank, the conjugate transpose of

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$A$, respectively. $\|\cdot\|_F$ and $\|\cdot\|_2$ stand for the matrix Frobenius norm and spectral norm, respectively, and $\|\cdot\|$ stands for the matrix Frobenius norm or spectral norm. We say a matrix is a *contraction* if its spectral norm is not greater than one.

For $A \in \mathbf{C}^{m\times n}$, let $X \in \mathbf{C}^{n\times m}$ satisfy some of the following four equations,

$$(1)\ AXA = A;\ \ (2)\ XAX = X;\ \ (3)\ (AX)^H = AX;\ \ (4)\ (XA)^H = XA. \tag{1.1}$$

Especially, if $X$ satisfies all of the above four equations, then $X$ is unique, called the Moore–Penrose inverse of $A$ and denoted by $X = A^\dagger$. If $X$ satisfies the first equation of (1.1), then $X$ is called a g-inverse of $A$ and denoted by $A^-$. It is well known that in general, g-inverses are not unique, and the set of all g-inverses of $A$ is denoted by $A\{1\}$.

We also denote the following orthogonal projections,

$$P_A = AA^\dagger,\quad P_A^\perp = I_m - AA^\dagger,\quad P_{A^H} = A^\dagger A,\quad P_{A^H}^\perp = I_n - A^\dagger A. \tag{1.2}$$

There have been much efforts to study error bounds of different generalized inverses and corresponding oblique projections. The perturbation analysis of Moore–Penrose inverses, orthogonal projections, and the least squares problems have been extensively studied, e.g., see [3,18,6,7,9,11,12,16,17, 19–22] and references cited therein. Wei [23], Li and Wei [13] derived perturbation bounds for the group inverse and corresponding oblique projection.

In this paper, we are interested in studying perturbation bounds of g-inverses and oblique projections. It is well known that the Moore–Penrose inverse, weighted Moore–Penrose inverse, {1, 3}-inverses, {1, 4}-inverses, and the group inverse all belong to g-inverses. Owning to the extensive applications in matrix theory and computation, g-inverses receive lots of consideration. Therefore, from perturbation analysis of g-inverses and oblique projections, we can better understand perturbation properties of specific g-inverses and oblique projections. Liu et al. [14] studied the continuity properties of g-inverses and oblique projections under rank invariant perturbations. To our knowledge, perturbation analysis for g-inverses and oblique projections have not been studied yet in the literature.

The paper is organized as follows. In Section 2, we briefly review some results for further discussions; in Section 3, for any given $A^- \in A\{1\}$ and $\widehat{A} = A + E$, we specify general formulas of the g-inverses $\widehat{A}^- \in \widehat{A}\{1\}$, such that $\widehat{A}^-$ is closest to $A^-$, $\widehat{A}\widehat{A}^-$ is closest to $AA^-$, or $\widehat{A}^-\widehat{A}$ is closest to $A^-A$ under appropriate norms; in Section 4, under an additional rank preserving condition $\mathrm{rank}(\widehat{A}) = \mathrm{rank}(A)$, we derive perturbation bounds of the nearest perturbed g-inverses and oblique projections for given g-inverse $A^-$ from the results obtained in Section 3; in Section 5, we provide perturbation bound for a consistent linear system; in Section 6, we report several numerical examples to verify the validity of our analysis; finally in Section 7, we make some concluding remarks.

## 2. Preliminaries

In this section, we mention the following results for our further discussions. The following two formulas are well known.

**Lemma 2.1** ([18]). *Suppose that* $A, \widehat{A} \in \mathbf{C}^{m\times n}, \widehat{A} = A + E$. *Then*

$$\begin{aligned}\widehat{A}^\dagger - A^\dagger &= -\widehat{A}^\dagger E A^\dagger + \widehat{A}^\dagger P_A^\perp - P_{\widehat{A}^H}^\perp A^\dagger\\ &= -\widehat{A}^\dagger P_{\widehat{A}} E P_{A^H} A^\dagger + \widehat{A}^\dagger P_{\widehat{A}} P_A^\perp - P_{\widehat{A}^H}^\perp P_{A^H} A^\dagger\\ &= -\widehat{A}^\dagger P_{\widehat{A}} E P_{A^H} A^\dagger + (\widehat{A}^H\widehat{A})^\dagger E^H P_A^\perp + P_{\widehat{A}^H}^\perp E^H (AA^H)^\dagger.\end{aligned} \tag{2.1}$$

**Lemma 2.2** ([2]). *For a matrix* $A \in \mathbf{C}^{m\times n}$, *any g-inverse* $A^-$ *of* $A$ *has the form*

$$\begin{aligned}A^- &= A^\dagger + Z - A^\dagger AZAA^\dagger\\ &= A^\dagger + A^\dagger AZ(I_m - AA^\dagger) + (I_n - A^\dagger A)Z\\ &= A^\dagger + (I_n - A^\dagger A)ZAA^\dagger + Z(I_m - AA^\dagger),\end{aligned} \tag{2.2}$$

*where* $Z \in \mathbf{C}^{n\times m}$ *is arbitrary.*

The following two lemmas are special cases of Davis–Kahan–Weinberger solutions of norm-preserving dilations, for more general cases of norm-preserving dilations we refer to [5].

**Lemma 2.3** ([5]). *For a given matrix $A \in \mathbf{C}^{m\times n}$ with $\|A\|_2 = \mu$, let*

$$Q = (\mu^2 I_n - A^H A)^{\frac{1}{2}}, \quad Q_* = (\mu^2 I_m - AA^H)^{\frac{1}{2}}. \tag{2.3}$$

*Then*

1. *There exists a matrix $B \in \mathbf{C}^{l\times n}$ such that*
$$\min_{B\in\mathbf{C}^{l\times n}} \left\| \begin{pmatrix} A \\ B \end{pmatrix} \right\|_2 = \mu, \tag{2.4}$$
*where $B$ has the form $B = KQ$ with $K \in \mathbf{C}^{l\times n}$ an arbitrary contraction.*
2. *There exists a matrix $C \in \mathbf{C}^{m\times k}$ such that*
$$\min_{C\in\mathbf{C}^{m\times k}} \|(A, C)\|_2 = \mu, \tag{2.5}$$
*where $C$ has the form $C = Q_* L$ with $L \in \mathbf{C}^{m\times k}$ an arbitrary contraction.*

**Lemma 2.4** (*[5]*). *Suppose that $A \in \mathbf{C}^{m\times n}$, $B \in \mathbf{C}^{p\times n}$, $C \in \mathbf{C}^{m\times q}$ satisfy*

$$\left\| \begin{pmatrix} A \\ B \end{pmatrix} \right\|_2 = \mu_1, \quad \|(A, C)\|_2 = \mu_2,$$

*and $\mu = \max\{\mu_1, \mu_2\}$. Then there exists $D \in \mathbf{C}^{p\times q}$ such that*

$$\min_{D\in\mathbf{C}^{p\times q}} \left\| \begin{pmatrix} A & C \\ B & D \end{pmatrix} \right\|_2 = \mu. \tag{2.6}$$

*Moreover, a general form of $D$ with this property is*

$$D = -KA^H L + \mu(I_p - KK^H)^{\frac{1}{2}} Z (I_q - L^H L)^{\frac{1}{2}}, \tag{2.7}$$

*where*

$$K^H = [(\mu^2 I_n - A^H A)^{\frac{1}{2}}]^\dagger B^H, \quad L = [(\mu^2 I_m - AA^H)^{\frac{1}{2}}]^\dagger C, \tag{2.8}$$

*and $Z \in \mathbf{C}^{p\times q}$ is an arbitrary contraction.*[1]

## 3. The nearest perturbed g-inverses and oblique projections

Liu et al. [14] proved that, for given matrices $A$, $\widehat{A} = A + E \in \mathbf{C}^{m\times n}$ with $\|E\|$ sufficiently small and $\operatorname{rank}(\widehat{A}) = \operatorname{rank}(A)$, then for any $A^- \in A\{1\}$, there exists a matrix $\widehat{A}^- \in \widehat{A}\{1\}$, such that $\|\widehat{A}^- - A^-\|$ is also small, and when $\|E\| \to 0$, $\|\widehat{A}^- - A^-\| \to 0$.

In this section, we will specify the formulas of $\widehat{A}^-$, such that for any $A^-$, $\|\widehat{A}^- - A^-\| = \min$, $\|\widehat{A}\widehat{A}^- - AA^-\| = \min$, or $\|\widehat{A}^-\widehat{A} - A^-A\| = \min$. In this section, we do not enforce the rank preserving condition $\operatorname{rank}(\widehat{A}) = \operatorname{rank}(A)$.

**Theorem 3.1.** *Suppose that $A, \widehat{A} \in \mathbf{C}^{m\times n}$. For any given $A^- \in A\{1\}$, there exists a unique matrix $\widehat{A}_m^- \in \widehat{A}\{1\}$ of the form*

$$\widehat{A}_m^- = \widehat{A}^\dagger + \widehat{A}^\dagger \widehat{A} A^- (I_m - \widehat{A}\widehat{A}^\dagger) + (I_n - \widehat{A}^\dagger \widehat{A}) A^-, \tag{3.1}$$

[1] In [5], the formulas in (2.8) is $K^H = [(\mu^2 I_n - A^H A)^{\frac{1}{2}}]^{-1} B^H$, $L = [(\mu^2 I_m - AA^H)^{\frac{1}{2}}]^{-1} C$. We use the formulas in (2.8) to cover the case $\mu = \|A\|_2$. A similar situation for norm preserving Hermitian matrix extension problem is studied in [10,24].

*such that*

$$\min_{\widehat{A}^-\in\widehat{A}\{1\}} \|\widehat{A}^- - A^-\|_F = \|\widehat{A}_m^- - A^-\|_F = \|\widehat{A}^\dagger - \widehat{A}^\dagger\widehat{A}A^-\widehat{A}\widehat{A}^\dagger\|_F. \tag{3.2}$$

**Proof.** From Lemma 2.2, any $\widehat{A}^-$ has the form

$$\widehat{A}^- = \widehat{A}^\dagger + \widehat{A}^\dagger\widehat{A}\widehat{Z}(I_m - \widehat{A}\widehat{A}^\dagger) + (I_n - \widehat{A}^\dagger\widehat{A})\widehat{Z}, \quad \widehat{Z} \in \mathbf{C}^{n\times m}. \tag{3.3}$$

Therefore,

$$\widehat{A}^- - A^- = (\widehat{A}^\dagger - \widehat{A}^\dagger\widehat{A}A^-\widehat{A}\widehat{A}^\dagger) + \widehat{A}^\dagger\widehat{A}(\widehat{Z} - A^-)(I_m - \widehat{A}\widehat{A}^\dagger) + (I_n - \widehat{A}^\dagger\widehat{A})(\widehat{Z} - A^-). \tag{3.4}$$

Since the three items in right-hand side of (3.4) are either row orthogonal or column orthogonal, by the property of Frobenius norm we have

$$\begin{aligned}\|\widehat{A}^- - A^-\|_F^2 &= \|\widehat{A}^\dagger - \widehat{A}^\dagger\widehat{A}A^-\widehat{A}\widehat{A}^\dagger\|_F^2 + \|\widehat{A}^\dagger\widehat{A}(\widehat{Z} - A^-)(I_m - \widehat{A}\widehat{A}^\dagger)\|_F^2 + \|(I_n - \widehat{A}^\dagger\widehat{A})(\widehat{Z} - A^-)\|_F^2\\ &\geqslant \|\widehat{A}^\dagger - \widehat{A}^\dagger\widehat{A}A^-\widehat{A}\widehat{A}^\dagger\|_F^2.\end{aligned}$$

The last inequality becomes an equality, if and only if

$$\widehat{A}^\dagger\widehat{A}\widehat{Z}(I_m - \widehat{A}\widehat{A}^\dagger) = \widehat{A}^\dagger\widehat{A}A^-(I_m - \widehat{A}\widehat{A}^\dagger), \quad (I_n - \widehat{A}^\dagger\widehat{A})\widehat{Z} = (I_n - \widehat{A}^\dagger\widehat{A})A^-,$$

proving the assertions of the theorem. □

From Theorem 3.1 we see that for given $A^-$, the nearest g-inverse $\widehat{A}_m^-$ is unique under the matrix Frobenius norm. However, the nearest g-inverse $\widehat{A}_m^-$ may not be unique under the spectral norm. To prove this, we first describe the singular value decomposition (SVD) of $\widehat{A}$. For $\widehat{A} \in \mathbf{C}_r^{m\times n}$, let

$$\widehat{A} = (\widehat{U}_1, \quad \widehat{U}_2)\,\mathrm{diag}\,(\widehat{\Sigma}, \quad 0)\,(\widehat{V}_1, \quad \widehat{V}_2)^H = \widehat{U}_1\widehat{\Sigma}\widehat{V}_1^H \tag{3.5}$$

be the SVD of $\widehat{A}$, where $\widehat{U}_1^H\widehat{U}_1 = \widehat{V}_1^H\widehat{V}_1 = I_r$ and $\widehat{\Sigma} = \mathrm{diag}(\widehat{\sigma}_1, \widehat{\sigma}_2, \cdots, \widehat{\sigma}_r) > 0$. Then we have the following relations

$$\begin{aligned}&\widehat{A}\widehat{A}^\dagger = \widehat{U}_1\widehat{U}_1^H, \quad I_m - \widehat{A}\widehat{A}^\dagger = \widehat{U}_2\widehat{U}_2^H,\\ &\widehat{A}^\dagger\widehat{A} = \widehat{V}_1\widehat{V}_1^H, \quad I_n - \widehat{A}^\dagger\widehat{A} = \widehat{V}_2\widehat{V}_2^H.\end{aligned} \tag{3.6}$$

We now have

**Theorem 3.2.** *Suppose that $A, \widehat{A} \in \mathbf{C}^{m\times n}$. For any given $A^- \in A\{1\}$, there exist matrices $\widehat{A}_m^- \in \widehat{A}\{1\}$, such that*

$$\min_{\widehat{A}^-\in\widehat{A}\{1\}} \|\widehat{A}^- - A^-\|_2 = \|\widehat{A}_m^- - A^-\|_2 = \|\widehat{A}^\dagger - \widehat{A}^\dagger\widehat{A}A^-\widehat{A}\widehat{A}^\dagger\|_2, \tag{3.7}$$

*and a general form of $\widehat{A}_m^-$ is*

$$\begin{aligned}\widehat{A}_m^- &= \widehat{A}^\dagger + \widehat{A}^\dagger\widehat{A}A^-(I_m - \widehat{A}\widehat{A}^\dagger) + (I_n - \widehat{A}^\dagger\widehat{A})A^-\\ &\quad + \widehat{A}^\dagger\widehat{A}(\mu^2 I_n - \mathcal{C}\mathcal{C}^H)^{\frac{1}{2}}\widehat{A}^\dagger\widehat{A}L(I_m - \widehat{A}\widehat{A}^\dagger) + (I_n - \widehat{A}^\dagger\widehat{A})K(\mu^2 I_m - \mathcal{A}^H\mathcal{A})^{\frac{1}{2}},\end{aligned} \tag{3.8}$$

*where $\mu = \|\widehat{A}^\dagger - \widehat{A}^\dagger\widehat{A}A^-\widehat{A}\widehat{A}^\dagger\|_2$,*

$$\begin{aligned}\mathcal{C} &= \widehat{A}^\dagger\widehat{A}(\widehat{A}^\dagger - A^-)\widehat{A}\widehat{A}^\dagger,\\ \mathcal{A} &= \widehat{A}^\dagger\widehat{A}(\widehat{A}^\dagger - A^-)\widehat{A}\widehat{A}^\dagger + \widehat{A}^\dagger\widehat{A}(\mu^2 I_n - \mathcal{C}\mathcal{C}^H)^{\frac{1}{2}}\widehat{A}^\dagger\widehat{A}L(I_m - \widehat{A}\widehat{A}^\dagger),\end{aligned} \tag{3.9}$$

*in which $K, L \in \mathbf{C}^{n\times m}$ are two arbitrary contractions.*

**Proof.** From the expression of $\widehat{A}^-$ in (3.3), we can rewrite $\widehat{A}^- - A^-$ as

$$\begin{aligned}\widehat{A}^- - A^- &= \widehat{A}^\dagger\widehat{A}[\widehat{A}^\dagger + \widehat{Z}(I_m - \widehat{A}\widehat{A}^\dagger) - A^-] + (I_n - \widehat{A}^\dagger\widehat{A})(\widehat{Z} - A^-)\\ &= \widehat{A}^\dagger\widehat{A}[\widehat{A}^\dagger + W(I_m - \widehat{A}\widehat{A}^\dagger) - A^-] + (I_n - \widehat{A}^\dagger\widehat{A})(\widehat{Z} - A^-).\end{aligned}$$

Here we can replace $\widehat{Z}$ in the first square bracket by an arbitrary matrix $W \in \mathbf{C}^{n\times m}$, because $\widehat{A}^{\dagger}\widehat{A}\widehat{Z}$ and $(I_n - \widehat{A}^{\dagger}\widehat{A})\widehat{Z}$ belong to two complementary projection spaces. Therefore from the SVD of $\widehat{A}$ in (3.5) we observe

$$\|\widehat{A}^- - A^-\|_2 = \left\|(\widehat{V}_1, \quad \widehat{V}_2)^H\,(\widehat{A}^- - A^-)\right\|_2 = \left\|\begin{pmatrix}\mathcal{A}_1\\ \mathcal{B}_1\end{pmatrix}\right\|_2,$$

where

$$\mathcal{A}_1 = \widehat{V}_1^H[\widehat{A}^\dagger + W(I_m - \widehat{A}\widehat{A}^\dagger) - A^-], \quad \mathcal{A} := \widehat{V}_1\mathcal{A}_1, \quad \mathcal{B}_1 = \widehat{V}_2^H(\widehat{Z} - A^-). \tag{3.10}$$

By applying Lemma 2.3, we observe

$$\min_{\widehat{Z}\in\mathbf{C}^{n\times m}} \|\widehat{A}^- - A^-\|_2 = \|\mathcal{A}_1\|_2 = \|\mathcal{A}\|_2 =: \mu_1$$

with the choice

$$\mathcal{B}_1 = \widehat{V}_2^H(\widehat{Z} - A^-) = K_1(\mu_1^2 I_m - \mathcal{A}_1^H\mathcal{A}_1)^{\frac{1}{2}} = K_1(\mu_1^2 I_m - \mathcal{A}^H\mathcal{A})^{\frac{1}{2}},$$

where $K_1 \in \mathbf{C}^{(n-r)\times m}$ is an arbitrary contraction. Therefore,

$$(I_n - \widehat{A}^\dagger\widehat{A})\widehat{Z} = \widehat{V}_2\widehat{V}_2^H\widehat{Z} = (I_n - \widehat{A}^\dagger\widehat{A})A^- + (I_n - \widehat{A}^\dagger\widehat{A})K(\mu_1^2 I_m - \mathcal{A}^H\mathcal{A})^{\frac{1}{2}}, \tag{3.11}$$

in which $K = \widehat{V}_2K_1$ is also a contraction. Furthermore,

$$\begin{aligned}\mu_1 = \|\mathcal{A}_1\|_2 = \|\mathcal{A}\|_2 &= \left\|\widehat{V}_1^H[\widehat{A}^\dagger + W(I_m - \widehat{A}\widehat{A}^\dagger) - A^-]\,(\widehat{U}_1, \quad \widehat{U}_2)\right\|_2\\ &= \|(\mathcal{C}_1, \quad \mathcal{D}_1)\|_2\end{aligned}$$

where

$$\mathcal{C}_1 = \widehat{V}_1^H(\widehat{A}^\dagger - A^-)\widehat{U}_1, \quad \mathcal{C} := \widehat{V}_1\mathcal{C}_1\widehat{U}_1^H, \quad \mathcal{D}_1 = \widehat{V}_1^H(W - A^-)\widehat{U}_2. \tag{3.12}$$

By applying Lemma 2.3 once again, we have

$$\min_{W\in\mathbf{C}^{n\times m}} \mu_1 = \|\mathcal{C}_1\|_2 = \|\mathcal{C}\|_2 = \|\widehat{A}^\dagger - \widehat{A}^\dagger\widehat{A}A^-\widehat{A}\widehat{A}^\dagger\|_2 =: \mu$$

with the choice

$$\mathcal{D}_1 = \widehat{V}_1^H(W - A^-)\widehat{U}_2 = (\mu^2 I_r - \mathcal{C}_1\mathcal{C}_1^H)^{\frac{1}{2}}L_1.$$

By simple derivation, we obtain

$$\begin{aligned}\widehat{A}^\dagger\widehat{A}(W - A^-)(I_m - \widehat{A}\widehat{A}^\dagger) &= \widehat{V}_1\mathcal{D}_1\widehat{U}_2^H = \widehat{V}_1(\mu^2 I_r - \mathcal{C}_1\mathcal{C}_1^H)^{\frac{1}{2}}L_1\widehat{U}_2^H\\ &= \widehat{A}^\dagger\widehat{A}(\mu^2 I_n - \mathcal{C}\mathcal{C}^H)^{\frac{1}{2}}\widehat{A}^\dagger\widehat{A}L(I_m - \widehat{A}\widehat{A}^\dagger),\end{aligned}$$

where $L_1 \in \mathbf{C}^{r\times(m-r)}$ is an arbitrary contraction, and so $L = \widehat{V}_1L_1\widehat{U}_2^H$ is also a contraction. Therefore,

$$\widehat{A}^\dagger\widehat{A}W(I_m - \widehat{A}\widehat{A}^\dagger) = \widehat{A}^\dagger\widehat{A}A^-(I_m - \widehat{A}\widehat{A}^\dagger) + \widehat{A}^\dagger\widehat{A}(\mu^2 I_n - \mathcal{C}\mathcal{C}^H)^{\frac{1}{2}}\widehat{A}^\dagger\widehat{A}L(I_m - \widehat{A}\widehat{A}^\dagger). \tag{3.13}$$

Substituting (3.13) into (3.10), we have

$$\mathcal{A} = \widehat{A}^\dagger\widehat{A}(\widehat{A}^\dagger - A^-)\widehat{A}\widehat{A}^\dagger + \widehat{A}^\dagger\widehat{A}(\mu^2 I_n - \mathcal{C}\mathcal{C}^H)^{\frac{1}{2}}\widehat{A}^\dagger\widehat{A}L(I_m - \widehat{A}\widehat{A}^\dagger), \tag{3.14}$$

and (3.11) is also updated by replacing $\mu_1$ with $\mu$. We then complete the proof of the theorem. $\square$

**Remark 3.1.** Notice that the g-inverse $\widehat{A}_m^-$ with the expression in (3.1) is a special g-inverse in (3.8) with $K = 0,\ L = 0$.

**Remark 3.2.** For the perturbation of the rank deficient least squares problem $\min\limits_{x\in\mathbf{C}^n}\|Ax - b\|_2$, $\min\limits_{\hat{x}\in\mathbf{C}^n}\|\widehat{A}x - \hat{b}\|_2$, where $A,\ \widehat{A} \in \mathbf{C}_r^{m\times n},\ b,\ \hat{b} \in \mathbf{C}^m$, Wei [20] proved that, for any LS solution

$$x = A^\dagger b + (I_n - A^\dagger A)z,$$

there exists a LS solution $\hat{x}$ of the form

$$\hat{x} = \widehat{A}^\dagger \hat{b} + (I_n - \widehat{A}^\dagger \widehat{A})(I_n - A^\dagger A)z,$$

such that $\|\hat{x} - x\|_2$ is small. Ding and Huang [8], Ding [7] then used an idea of the orthogonal projection of a point onto a linear manifold, obtained that with the LS solution of the perturbed LS problem,

$$\hat{x}_m = \widehat{A}^\dagger \hat{b} + (I_n - \widehat{A}^\dagger \widehat{A})[A^\dagger b + (I_n - A^\dagger A)z],$$

the following formula holds:

$$\|\hat{x}_m - x\|_2 = \|\widehat{A}^\dagger \hat{b} - \widehat{A}^\dagger \widehat{A}[A^\dagger b + (I_n - A^\dagger A)z]\|_2 = \min_{\hat{x}} \|\hat{x} - x\|_2.$$

By applying the idea of proving Theorems 3.1 and 3.2, we can study the following rank deficient LS problems, in which the observation vector is replaced by a matrix.

**Theorem 3.3.** *Suppose that $A, \widehat{A} \in \mathbf{C}^{m\times n}$ and $B, \widehat{B} = B + F \in \mathbf{C}^{m\times d}$. Consider the following LS problems*

$$\begin{aligned} S &= \{X : \|AX - B\|_{\mathrm{F}} = \min_{Y\in\mathbf{C}^{n\times d}} \|AY - B\|_{\mathrm{F}}\}, \\ \widehat{S} &= \{\widehat{X} : \|\widehat{A}\widehat{X} - \widehat{B}\|_{\mathrm{F}} = \min_{Y\in\mathbf{C}^{n\times d}} \|\widehat{A}Y - \widehat{B}\|_{\mathrm{F}}\}. \end{aligned} \tag{3.15}$$

*Then for any LS solution $X \in S$ of the form*

$$X = A^\dagger B + (I_n - A^\dagger A)Z, \quad Z \in \mathbf{C}^{n\times d}, \tag{3.16}$$

*there exists a unique LS solution $\widehat{X}_m \in \widehat{S}$ of the form*

$$\widehat{X}_m = \widehat{A}^\dagger \widehat{B} + (I_n - \widehat{A}^\dagger \widehat{A})X = \widehat{A}^\dagger \widehat{B} + (I_n - \widehat{A}^\dagger \widehat{A})[A^\dagger B + (I_n - A^\dagger A)Z], \tag{3.17}$$

*such that*

$$\|\widehat{X}_m - X\|_{\mathrm{F}} = \min_{\widehat{X}\in\widehat{S}} \|\widehat{X} - X\|_{\mathrm{F}} = \|\widehat{A}^\dagger \widehat{B} - \widehat{A}^\dagger \widehat{A}X\|_{\mathrm{F}}. \tag{3.18}$$

*Also, there exists a LS solution $\widehat{X}_m$, such that*

$$\|\widehat{X}_m - X\|_2 = \min_{\widehat{X}\in\widehat{S}} \|\widehat{X} - X\|_2 = \|\widehat{A}^\dagger \widehat{B} - \widehat{A}^\dagger \widehat{A}X\|_2 =: \mu, \tag{3.19}$$

*and a general form of $\widehat{X}_m$ satisfying (3.19) is*

$$\widehat{X}_m = \widehat{A}^\dagger \widehat{B} + (I_n - \widehat{A}^\dagger \widehat{A})X + (I_n - \widehat{A}^\dagger \widehat{A})K(\mu^2 I_d - \mathcal{A}^H \mathcal{A})^{\frac{1}{2}}, \tag{3.20}$$

*where $\mathcal{A} = \widehat{A}^\dagger \widehat{B} - \widehat{A}^\dagger \widehat{A}X$, and $K \in \mathbf{C}^{n\times d}$ is an arbitrary contraction.*

**Proof.** From the formulas of $\widehat{X}$ and $X$, we have

$$\begin{aligned} \|\widehat{X} - X\|_{\mathrm{F}}^2 &= \|(\widehat{A}^\dagger \widehat{B} - \widehat{A}^\dagger \widehat{A}X) + (I_n - \widehat{A}^\dagger \widehat{A})(\widehat{Z} - X)\|_{\mathrm{F}}^2 \\ &= \|\widehat{A}^\dagger \widehat{B} - \widehat{A}^\dagger \widehat{A}X\|_{\mathrm{F}}^2 + \|(I_n - \widehat{A}^\dagger \widehat{A})(\widehat{Z} - X)\|_{\mathrm{F}}^2 \\ &\geqslant \|\widehat{A}^\dagger \widehat{B} - \widehat{A}^\dagger \widehat{A}X\|_{\mathrm{F}}^2, \end{aligned}$$

and the last inequality becomes an equality if and only if

$$(I_n - \widehat{A}^\dagger \widehat{A})\widehat{Z} = (I_n - \widehat{A}^\dagger \widehat{A})X,$$

therefore the formulas in (3.17) and (3.18) hold.

From the SVD of $\widehat{A}$ in (3.5), we have

$$\|\widehat{X} - X\|_2 = \left\|(\widehat{V}_1, \quad \widehat{V}_2)^H (\widehat{X} - X)\right\|_2 = \left\|\begin{pmatrix}\mathcal{A}_1 \\ \mathcal{B}_1\end{pmatrix}\right\|_2,$$

where

$$\mathcal{A}_1 = \widehat{V}_1^H(\widehat{A}^\dagger\widehat{B} - X), \quad \mathcal{A} := \widehat{V}_1\mathcal{A}_1, \quad \mathcal{B}_1 = \widehat{V}_2^H(\widehat{Z} - X). \tag{3.21}$$

By applying Lemma 2.3, we observe

$$\min_{\widehat{X}\in\widehat{S}} \|\widehat{X} - X\|_2 = \|\mathcal{A}_1\|_2 = \|\mathcal{A}\|_2 = \mu$$

with the choice

$$\mathcal{B}_1 = \widehat{V}_2^H(\widehat{Z} - X) = K_1(\mu^2 I_d - \mathcal{A}_1^H\mathcal{A}_1)^{\frac{1}{2}} = K_1(\mu^2 I_d - \mathcal{A}^H\mathcal{A})^{\frac{1}{2}},$$

where $K_1 \in \mathbf{C}^{(n-r)\times d}$ is an arbitrary contraction. Therefore,

$$(I_n - \widehat{A}^\dagger\widehat{A})\widehat{Z} = \widehat{V}_2\widehat{V}_2^H\widehat{Z} = (I_n - \widehat{A}^\dagger\widehat{A})X + (I_n - \widehat{A}^\dagger\widehat{A})K(\mu^2 I_d - \mathcal{A}^H\mathcal{A})^{\frac{1}{2}}, \tag{3.22}$$

in which $K = \widehat{V}_2K_1$ is also a contraction. Therefore, the formulas in (3.19) and (3.20) also hold. We then complete the proof of the theorem. $\square$

We now derive g-inverses of the perturbation matrix such that the oblique projections are the nearest under the matrix Frobenius norm or spectral norm.

**Theorem 3.4.** *Suppose that $A, \widehat{A} \in \mathbf{C}^{m\times n}$. For any given $A^- \in A\{1\}$, there exists a matrix $\widehat{A}_p^- \in \widehat{A}\{1\}$ of the form*

$$\widehat{A}_p^- = \widehat{A}^\dagger + \widehat{A}^\dagger AA^-(I_m - \widehat{A}\widehat{A}^\dagger) + (I_n - \widehat{A}^\dagger\widehat{A})\widehat{Z}, \tag{3.23}$$

*in which $\widehat{Z} \in \mathbf{C}^{n\times m}$ is arbitrary, such that*

$$\begin{aligned}\min_{\widehat{A}^-\in\widehat{A}\{1\}} ||\widehat{A}\widehat{A}^- - AA^-||_{\mathrm{F}} &= ||\widehat{A}\widehat{A}_p^- - AA^-||_{\mathrm{F}} \\ &= ||\widehat{A}\widehat{A}^\dagger(I_m - AA^-)\widehat{A}\widehat{A}^\dagger + (I_m - \widehat{A}\widehat{A}^\dagger)AA^-||_{\mathrm{F}}.\end{aligned} \tag{3.24}$$

**Proof.** Using the formula of $\widehat{A}^-$ in (3.3), we have

$$\widehat{A}\widehat{A}^- = \widehat{A}\widehat{A}^\dagger + \widehat{A}\widehat{Z}(I_m - \widehat{A}\widehat{A}^\dagger). \tag{3.25}$$

Then we have

$$\widehat{A}\widehat{A}^- - AA^- = \widehat{A}\widehat{A}^\dagger(I_m - AA^-)\widehat{A}\widehat{A}^\dagger - (I_m - \widehat{A}\widehat{A}^\dagger)AA^- + \widehat{A}\widehat{A}^\dagger(\widehat{A}\widehat{Z} - AA^-)(I_m - \widehat{A}\widehat{A}^\dagger). \tag{3.26}$$

Thus

$$\begin{aligned}||\widehat{A}\widehat{A}^- - AA^-||_{\mathrm{F}}^2 &= ||\widehat{A}\widehat{A}^\dagger(I_m - AA^-)\widehat{A}\widehat{A}^\dagger||_{\mathrm{F}}^2 + ||(I_m - \widehat{A}\widehat{A}^\dagger)AA^-||_{\mathrm{F}}^2 \\ &\quad + ||\widehat{A}\widehat{A}^\dagger(\widehat{A}\widehat{Z} - AA^-)(I_m - \widehat{A}\widehat{A}^\dagger)||_{\mathrm{F}}^2 \\ &\geqslant ||\widehat{A}\widehat{A}^\dagger(I_m - AA^-)\widehat{A}\widehat{A}^\dagger||_{\mathrm{F}}^2 + ||(I_m - \widehat{A}\widehat{A}^\dagger)AA^-||_{\mathrm{F}}^2 \\ &= ||\widehat{A}\widehat{A}^\dagger(I_m - AA^-)\widehat{A}\widehat{A}^\dagger + (I_m - \widehat{A}\widehat{A}^\dagger)AA^-||_{\mathrm{F}}^2,\end{aligned}$$

where the inequality becomes an equality if and only if

$$\widehat{A}\widehat{Z}(I_m - \widehat{A}\widehat{A}^\dagger) = \widehat{A}\widehat{A}^\dagger AA^-(I_m - \widehat{A}\widehat{A}^\dagger),$$

and so

$$\widehat{A}_p^- = \widehat{A}^\dagger + \widehat{A}^\dagger AA^-(I_m - \widehat{A}\widehat{A}^\dagger) + (I_n - \widehat{A}^\dagger\widehat{A})\widehat{Z},$$

where $\widehat{Z} \in \mathbf{C}^{n\times m}$ is arbitrary, and so we complete the proof of the theorem. $\square$

Similarly, applying the last equality in (2.2), we obtain the following result.

**Theorem 3.5.** *Suppose that* $A, \widehat{A} \in \mathbf{C}^{m\times n}$. *For any given* $A^- \in A\{1\}$, *there exists a matrix* $\widehat{A}_p^- \in \widehat{A}\{1\}$ *of the form*

$$\widehat{A}_p^- = \widehat{A}^\dagger + (I_n - \widehat{A}^\dagger\widehat{A})A^-A\widehat{A}^\dagger + \widehat{Z}(I_m - \widehat{A}\widehat{A}^\dagger), \tag{3.27}$$

*in which* $\widehat{Z} \in \mathbf{C}^{n\times m}$ *is arbitrary, such that*

$$\begin{aligned}\min_{\widehat{A}^-\in\widehat{A}\{1\}} ||\widehat{A}^-\widehat{A} - A^-A||_{\mathrm{F}} &= ||\widehat{A}_p^-\widehat{A} - A^-A||_{\mathrm{F}} \\ &= ||\widehat{A}^\dagger\widehat{A}(I_n - A^-A)\widehat{A}^\dagger\widehat{A} + A^-A(I_n - \widehat{A}^\dagger\widehat{A})||_{\mathrm{F}}.\end{aligned} \tag{3.28}$$

**Theorem 3.6.** *Suppose that* $A, \widehat{A} \in \mathbf{C}^{m\times n}$. *For any given* $A^- \in A\{1\}$, *there exists a matrix* $\widehat{A}_p^- \in \widehat{A}\{1\}$, *such that*

$$\min_{\widehat{A}^-\in\widehat{A}\{1\}} ||\widehat{A}\widehat{A}^- - AA^-||_2 = ||\widehat{A}\widehat{A}_p^- - AA^-||_2 = \mu, \tag{3.29}$$

*where* $\mu = \max\{\|(I_m - AA^-)\widehat{A}\widehat{A}^\dagger\|_2,\ \|(I_m - \widehat{A}\widehat{A}^\dagger)AA^-\|_2\}$, *and a general form of* $\widehat{A}_p^-$ *is*

$$\begin{aligned}\widehat{A}_p^- &= \widehat{A}^\dagger + \widehat{A}^\dagger AA^-(I_m - \widehat{A}\widehat{A}^\dagger) + (I_n - \widehat{A}^\dagger\widehat{A})\widehat{Z} - \widehat{A}^\dagger K\mathcal{A}^HL \\ &\quad + \ \mu\widehat{A}^\dagger(I_m - KK^H)^{\frac{1}{2}}\widehat{A}\widehat{A}^\dagger W(I_m - \widehat{A}\widehat{A}^\dagger)(I_m - L^HL)^{\frac{1}{2}}(I_m - \widehat{A}\widehat{A}^\dagger),\end{aligned} \tag{3.30}$$

*in which*

$$\begin{aligned}&\mathcal{A} = -(I_m - \widehat{A}\widehat{A}^\dagger)AA^-\widehat{A}\widehat{A}^\dagger, \quad \mathcal{B} = \widehat{A}\widehat{A}^\dagger(I_m - AA^-)\widehat{A}\widehat{A}^\dagger, \\ &\mathcal{C} = -(I_m - \widehat{A}\widehat{A}^\dagger)AA^-(I_m - \widehat{A}\widehat{A}^\dagger), \\ &K^H = \widehat{A}\widehat{A}^\dagger[(\mu^2I_m - \mathcal{A}^H\mathcal{A})^{\frac{1}{2}}]^\dagger\mathcal{B}^H, \quad L = (I_m - \widehat{A}\widehat{A}^\dagger)[(\mu^2I_m - \mathcal{A}\mathcal{A}^H)^{\frac{1}{2}}]^\dagger\mathcal{C},\end{aligned}$$

$W \in \mathbf{C}^{m\times m}$ *is an arbitrary contraction, and* $\widehat{Z} \in \mathbf{C}^{n\times m}$ *is an arbitrary matrix*.

**Proof.** From the equality (3.26), we have

$$\widehat{A}\widehat{A}^- - AA^- = \widehat{A}\widehat{A}^\dagger(I_m - AA^-)\widehat{A}\widehat{A}^\dagger - (I_m - \widehat{A}\widehat{A}^\dagger)AA^- + \widehat{A}\widehat{A}^\dagger(\widehat{A}\widehat{Z} - AA^-)(I_m - \widehat{A}\widehat{A}^\dagger). \tag{3.31}$$

Therefore, by applying the SVD of $\widehat{A}$ in (3.5), we observe that

$$\|\widehat{A}\widehat{A}^- - AA^-\|_2 = \left\|(\widehat{U}_2, \widehat{U}_1\ )^H(\widehat{A}\widehat{A}^- - AA^-)(\widehat{U}_1, \quad \widehat{U}_2)\right\|_2 = \left\|\begin{pmatrix}\mathcal{A}_1 & \mathcal{C}_1 \\ \mathcal{B}_1 & \mathcal{D}_1\end{pmatrix}\right\|_2,$$

in which

$$\begin{aligned}&\mathcal{A}_1 = -\widehat{U}_2^HAA^-\widehat{U}_1, \quad \mathcal{C}_1 = -\widehat{U}_2^HAA^-\widehat{U}_2, \\ &\mathcal{B}_1 = \widehat{U}_1^H(I_m - AA^-)\widehat{U}_1, \quad \mathcal{D}_1 = \widehat{U}_1^H(\widehat{A}\widehat{Z} - AA^-)\widehat{U}_2.\end{aligned}$$

By applying Lemma 2.4 with the related footnote under consideration, we observe

$$\min_{\widehat{Z}\in\mathbf{C}^{n\times m}} \|\widehat{A}\widehat{A}^- - AA^-\|_2 = \|\widehat{A}\widehat{A}_p^- - AA^-\|_2 = \max\{\mu_1,\ \mu_2\}, \tag{3.32}$$

where

$$\begin{aligned}\mu_1 &= \left\|\begin{pmatrix}\mathcal{A}_1 \\ \mathcal{B}_1\end{pmatrix}\right\|_2 = \left\|(\widehat{U}_2, \quad \widehat{U}_1)\begin{pmatrix}\mathcal{A}_1 \\ \mathcal{B}_1\end{pmatrix}\widehat{U}_1^H\right\|_2 \\ &= \|\widehat{A}\widehat{A}^\dagger(I_m - AA^-)\widehat{A}\widehat{A}^\dagger - (I_m - \widehat{A}\widehat{A}^\dagger)AA^-\widehat{A}\widehat{A}^\dagger\|_2 \\ &= \|(I_m - AA^-)\widehat{A}\widehat{A}^\dagger\|_2,\end{aligned}$$

$$\begin{aligned}\mu_2 &= \|(\mathcal{A}_1, \quad \mathcal{C}_1)\|_2 = \left\|\widehat{U}_2(\mathcal{A}_1, \quad \mathcal{C}_1)(\widehat{U}_1, \quad \widehat{U}_2)^H\right\|_2 \\ &= \left\|(I_m - \widehat{A}\widehat{A}^\dagger)AA^-\widehat{A}\widehat{A}^\dagger + (I_m - \widehat{A}\widehat{A}^\dagger)AA^-(I_m - \widehat{A}\widehat{A}^\dagger)\right\|_2 \\ &= \|(I_m - \widehat{A}\widehat{A}^\dagger)AA^-\|_2,\end{aligned}$$

with the choice

$$\mathcal{D}_1 = \widehat{U}_1^H(\widehat{A}\widehat{Z} - AA^-)\widehat{U}_2 = -K_1\mathcal{A}_1^H L_1 + \mu(I_r - K_1K_1^H)^{\frac{1}{2}} W_1(I_{m-r} - L_1^H L_1)^{\frac{1}{2}},$$

where $W_1 \in \mathbf{C}^{r\times(m-r)}$ is an arbitrary contraction, and

$$K_1^H = [(\mu^2 I_r - \mathcal{A}_1^H\mathcal{A}_1)^{\frac{1}{2}}]^\dagger \mathcal{B}_1^H, \quad L_1 = [(\mu^2 I_{m-r} - \mathcal{A}_1\mathcal{A}_1^H)^{\frac{1}{2}}]^\dagger \mathcal{C}_1.$$

Setting

$$K = \widehat{U}_1 K_1 \widehat{U}_1^H, \quad L = \widehat{U}_2 L_1 \widehat{U}_2^H,$$

and

$$\mathcal{A} = \widehat{U}_2\mathcal{A}_1\widehat{U}_1^H,\ \mathcal{B} = \widehat{U}_1\mathcal{B}_1\widehat{U}_1^H,\ \mathcal{C} = \widehat{U}_2\mathcal{C}_1\widehat{U}_2^H.$$

By simple derivation, we have

$$\begin{aligned}
&\widehat{A}\widehat{A}^\dagger(\widehat{A}\widehat{Z} - AA^-)(I_m - \widehat{A}\widehat{A}^\dagger)\\
&\quad = \widehat{U}_1\mathcal{D}_1\widehat{U}_2^H\\
&\quad = -\widehat{U}_1K_1\mathcal{A}_1^H L_1\widehat{U}_2^H + \mu\widehat{U}_1(I_r - K_1K_1^H)^{\frac{1}{2}} W_1(I_{m-r} - L_1^H L_1)^{\frac{1}{2}}\widehat{U}_2^H\\
&\quad = -K\mathcal{A}^H L + \mu\widehat{A}\widehat{A}^\dagger(I_m - KK^H)^{\frac{1}{2}}\widehat{A}\widehat{A}^\dagger W(I_m - \widehat{A}\widehat{A}^\dagger)(I_m - L^H L)^{\frac{1}{2}}(I_m - \widehat{A}\widehat{A}^\dagger),
\end{aligned}$$

where $W = \widehat{U}_1 W_1 \widehat{U}_2^H \in \mathbf{C}^{m\times m}$ is also a contraction. Therefore

$$\begin{aligned}
\widehat{A}\widehat{Z}(I_m - \widehat{A}\widehat{A}^\dagger) = {}& \widehat{A}\widehat{A}^\dagger AA^-(I_m - \widehat{A}\widehat{A}^\dagger) - K\mathcal{A}^H L\\
&+ \mu\widehat{A}\widehat{A}^\dagger(I_m - KK^H)^{\frac{1}{2}}\widehat{A}\widehat{A}^\dagger W(I_m - \widehat{A}\widehat{A}^\dagger)(I_m - L^H L)^{\frac{1}{2}}(I_m - \widehat{A}\widehat{A}^\dagger).
\end{aligned}$$

By substituting the above formula into (3.3), we obtain the formula for $\widehat{A}_p^-$ in (3.30), and the equality in (3.29) also holds. We then complete the proof of the theorem. □

Similarly, for another oblique projection, we have the following result.

**Theorem 3.7.** *Suppose that $A, \widehat{A} \in \mathbf{C}^{m\times n}$. For any given $A^- \in A\{1\}$, there exists a matrix $\widehat{A}_p^- \in \widehat{A}\{1\}$ such that*

$$\min_{\widehat{A}^-\in\widehat{A}\{1\}} ||\widehat{A}^-\widehat{A} - A^-A||_2 = ||\widehat{A}_p^-\widehat{A} - A^-A||_2 = \mu, \tag{3.33}$$

*where $\mu = \max\{\|\widehat{A}^\dagger\widehat{A}(I_n - A^-A)\|_2,\ \|A^-A(I_n - \widehat{A}^\dagger\widehat{A})\|_2\}$, and a general form of $\widehat{A}_p^-$ is*

$$\begin{aligned}
\widehat{A}_p^- = {}& \widehat{A}^\dagger + (I_n - \widehat{A}^\dagger\widehat{A})A^-A\widehat{A}^\dagger + \widehat{Z}(I_m - \widehat{A}\widehat{A}^\dagger) - K\mathcal{A}^H L\widehat{A}^\dagger\\
&+ \mu(I_n - \widehat{A}^\dagger\widehat{A})(I_n - KK^H)^{\frac{1}{2}}(I_n - \widehat{A}^\dagger\widehat{A})W\widehat{A}^\dagger\widehat{A}(I_n - L^H L)^{\frac{1}{2}}\widehat{A}^\dagger,
\end{aligned} \tag{3.34}$$

*where*

$$\begin{aligned}
&\mathcal{A} = -\widehat{A}^\dagger\widehat{A}A^-A(I_n - \widehat{A}^\dagger\widehat{A}), \quad \mathcal{B} = -(I_n - \widehat{A}^\dagger\widehat{A})A^-A(I_n - \widehat{A}^\dagger A),\\
&\mathcal{C} = \widehat{A}^\dagger\widehat{A}(I_n - A^-A)\widehat{A}^\dagger\widehat{A},\\
&K^H = (I_n - \widehat{A}^\dagger A)[(\mu^2 I_n - \mathcal{A}^H\mathcal{A})^{\frac{1}{2}}]^\dagger\mathcal{B}^H, \quad L = \widehat{A}^\dagger\widehat{A}[(\mu^2 I_n - \mathcal{A}\mathcal{A}^H)^{\frac{1}{2}}]^\dagger\mathcal{C},
\end{aligned}$$

*$W \in \mathbf{C}^{n\times n}$ is an arbitrary contraction, and $\widehat{Z} \in \mathbf{C}^{n\times m}$ is an arbitrary matrix.*

**Remark 3.3.** It is worthy to point out that, for the perturbation of the orthogonal projections, we have [16]

$$||\widehat{A}\widehat{A}^\dagger - AA^\dagger||_2 = \max\left\{\|(I_m - AA^\dagger)\widehat{A}\widehat{A}^\dagger\|_2,\ \|(I_m - \widehat{A}\widehat{A}^\dagger)AA^\dagger\|_2\right\},$$

and from (3.29), we also have

$$\min_{\widehat{A}^-\in\widehat{A}\{1\}} ||\widehat{A}\widehat{A}^- - AA^\dagger||_2 = \max\left\{\|(I_m - AA^\dagger)\widehat{A}\widehat{A}^\dagger\|_2,\ \|(I_m - \widehat{A}\widehat{A}^\dagger)AA^\dagger\|_2\right\}.$$

## 4. Perturbation bounds for the nearest perturbed g-inverses and oblique projections

In this section, we derive the perturbation bounds for the nearest perturbed g-inverses and oblique projections by using the results obtained in the previous section.

Notice that in the analysis of the previous section, we do not enforce the condition $\mathrm{rank}(\widehat{A}) = \mathrm{rank}(A)$. Liu et al. [14] proved that, for stable perturbations the condition $\mathrm{rank}(\widehat{A}) = \mathrm{rank}(A)$ is necessary. This fact can be easily proven as follows. Suppose that $A, \widehat{A} = A + E \in \mathbf{C}^{m\times n}$ with $\mathrm{rank}(\widehat{A}) > \mathrm{rank}(A) = r$ and $\|E\|_2 \ll 1$. Then for given $A^- \in A\{1\}$ and any $\widehat{A}^- \in \widehat{A}\{1\}$, because $\|\widehat{A}^-\|_2 \geqslant \|\widehat{A}^\dagger\|_2$, we have from [15,16] that, when $\|E\|_2 \to 0$,

$$\begin{aligned}&\|\widehat{A}^-\|_2 \geqslant \|\widehat{A}^\dagger\|_2 \geqslant \tfrac{1}{\|E\|_2} \to \infty,\\ &\|\widehat{A}^- - A^-\|_2 \geqslant \|\widehat{A}^-\|_2 - \|A^-\|_2 \geqslant \|\widehat{A}^\dagger\|_2 - \|A^-\|_2 \geqslant \tfrac{1}{\|E\|_2} - \|A^-\|_2 \to \infty.\end{aligned}$$

**Theorem 4.1.** *Suppose that $A, \widehat{A} = A + E \in \mathbf{C}^{m\times n}$ with $\mathrm{rank}(A) = \mathrm{rank}(\widehat{A}) = r > 0$ and $\|A^\dagger\|_2\|E\|_2 < 1$. For any given $A^- \in A\{1\}$ of the form*

$$A^- = A^\dagger + A^\dagger AZ(I_m - AA^\dagger) + (I_n - A^\dagger A)Z, \quad Z \in \mathbf{C}^{n\times m}, \tag{4.1}$$

*let $\widehat{A}_m^-$ be as in Theorem 3.1 or 3.2. Then we have the following estimates,*

$$\begin{aligned}||\widehat{A}_m^- - A^-|| &\leqslant ||A^\dagger||_2(||A^\dagger + (I_n - A^\dagger A)Z||||P_{\widehat{A}}E||_2\\ &\quad + ||A^\dagger AZ(I_m - AA^\dagger)||||P_A^\perp EP_{\widehat{A}^H}||_2) + \mathcal{O}(||E||||E||_2),\\ \tfrac{||\widehat{A}_m^- - A^-||}{\|A^-\|} &\leqslant ||A^\dagger||_2(||P_{\widehat{A}}E||_2 + ||P_A^\perp EP_{\widehat{A}^H}||_2) + \mathcal{O}(||E||||E||_2),\end{aligned} \tag{4.2}$$

*where $||\cdot||$ is either the matrix Frobenius norm or spectral norm.*

**Proof.** We have from (3.2) and (3.7) that

$$\begin{aligned}&\|\widehat{A}_m^- - A^-\|\\ &\quad = \|\widehat{A}^\dagger - \widehat{A}^\dagger\widehat{A}A^-\widehat{A}\widehat{A}^\dagger\|\\ &\quad = \|\widehat{A}^\dagger\widehat{A}[\widehat{A}^\dagger - A^\dagger - A^\dagger AZ(I_m - AA^\dagger) - (I_n - A^\dagger A)Z]\widehat{A}\widehat{A}^\dagger\|\\ &\quad = \|\widehat{A}^\dagger\widehat{A}[-\widehat{A}^\dagger EA^\dagger + \widehat{A}^\dagger(I_m - AA^\dagger) - A^\dagger AZ(I_m - AA^\dagger) - (I_n - A^\dagger A)Z]\widehat{A}\widehat{A}^\dagger\|\\ &\quad \leqslant ||\widehat{A}^\dagger||_2(||A^\dagger + (I_n - A^\dagger A)Z||||P_{\widehat{A}}E||_2 + ||A^\dagger AZ(I_m - AA^\dagger)||||P_A^\perp EP_{\widehat{A}^H}||_2)\\ &\qquad + ||\widehat{A}^\dagger||_2^3||P_A^\perp EP_{\widehat{A}^H}||_2||P_A^\perp EP_{\widehat{A}^H}||.\end{aligned} \tag{4.3}$$

Also from the conditions of the theorem, we observe

$$\begin{aligned}&\tfrac{\|A^\dagger\|_2}{1+\|E\|_2\|A^\dagger\|_2} \leqslant \|\widehat{A}^\dagger\|_2 \leqslant \tfrac{\|A^\dagger\|_2}{1-\|E\|_2\|A^\dagger\|_2},\\ &||A^\dagger + (I_n - A^\dagger A)Z|| \leqslant \|A^-\|,\ ||A^\dagger AZ(I_m - AA^\dagger)|| \leqslant \|A^-\|,\\ &||AZ(I_m - AA^\dagger)|| \leqslant \|AA^-\|,\end{aligned} \tag{4.4}$$

from which, the inequalities of (4.2) follow. □

The following theorems elaborate the perturbation bounds for the nearest perturbed oblique projections.

**Theorem 4.2.** *Suppose that $A, \widehat{A} = A + E \in \mathbf{C}^{m\times n}$ with $\mathrm{rank}(A) = \mathrm{rank}(\widehat{A}) = r > 0$ and $\|A^\dagger\|_2\|E\|_2 < 1$. For any given $A^- \in A\{1\}$ of the form*

$$A^- = A^\dagger + A^\dagger AZ(I_m - AA^\dagger) + (I_n - A^\dagger A)Z, \quad Z \in \mathbf{C}^{n\times m} \tag{4.5}$$

*let $\widehat{A}_p^-$ be as in Theorem 3.4. Then we have the following estimates,*

$$\begin{aligned}||\widehat{A}\widehat{A}_p^- - AA^-||_F &\leqslant ||A^\dagger||_2(||AZ(I_m - AA^\dagger)||_F||P_A^\perp EP_{\widehat{A}^H}||_2\\ &\quad + ||AA^-||_F||P_{\widehat{A}}^\perp EP_{A^H}||_2) + \mathcal{O}(||E||_F||E||_2),\\ \tfrac{||\widehat{A}\widehat{A}_p^- - AA^-||_F}{||AA^-||_F} &\leqslant ||A^\dagger||_2(||P_A^\perp EP_{\widehat{A}^H}||_2 + ||P_{\widehat{A}}^\perp EP_{A^H}||_2) + \mathcal{O}(||E||_F||E||_2).\end{aligned} \tag{4.6}$$

**Proof.** From (3.24) we have

$$||\widehat{A}\widehat{A}_p^- - AA^-||_F \leqslant ||\widehat{A}\widehat{A}^\dagger(I_m - AA^-)\widehat{A}\widehat{A}^\dagger||_F + ||(I_m - \widehat{A}\widehat{A}^\dagger)EA^\dagger AA^-||_F. \tag{4.7}$$

Since

$$\begin{aligned}
&||\widehat{A}\widehat{A}^\dagger(I_m - AA^-)\widehat{A}\widehat{A}^\dagger||_F \\
&\quad = ||\widehat{A}\widehat{A}^\dagger[I_m - AA^\dagger - AZ(I_m - AA^\dagger)]\widehat{A}\widehat{A}^\dagger||_F \\
&\quad \leqslant ||\widehat{A}\widehat{A}^\dagger AZ(I_m - AA^\dagger)E\widehat{A}^\dagger||_F + ||(\widehat{A}^\dagger)^H E^H (I_m - AA^\dagger)E\widehat{A}^\dagger||_F \\
&\quad \leqslant ||\widehat{A}^\dagger||_2||AZ(I_m - AA^\dagger)||_F||P_A^\perp EP_{\widehat{A}^H}||_2 + ||\widehat{A}^\dagger||_2^2||P_A^\perp EP_{\widehat{A}^H}||_F||P_A^\perp EP_{\widehat{A}^H}||_2, \\
&||(I_m - \widehat{A}\widehat{A}^\dagger)EA^\dagger AA^-||_F \leqslant \|A^\dagger||_2||P_{\widehat{A}}^\perp EP_{A^H}\|_2\|AA^-||_F,
\end{aligned}$$

therefore by applying the inequalities in (4.4), we complete the proof of the theorem. □

Similarly, we have the following perturbation bound for the nearest perturbed oblique projection $A^-A$.

**Theorem 4.3.** *Suppose that $A, \widehat{A} = A + E \in \mathbf{C}^{m\times n}$ with $\operatorname{rank}(A) = \operatorname{rank}(\widehat{A}) = r > 0$ and $\|A^\dagger\|_2\|E\|_2 < 1$. For any given $A^- \in A\{1\}$ of the form*

$$A^- = A^\dagger + (I_n - A^\dagger A)ZAA^\dagger + Z(I_m - AA^\dagger), \quad Z \in \mathbf{C}^{n\times m} \tag{4.8}$$

*let $\widehat{A}_p^- \in \widehat{A}\{1\}$ be as in Theorem 3.5. Then we have the following estimates,*

$$\begin{aligned}
||\widehat{A}_p^-\widehat{A} - A^-A||_F &\leqslant ||A^\dagger||_2(||(I_n - A^\dagger A)ZA||_F||P_{\widehat{A}}EP_{A^H}^\perp||_2 + ||A^-A||_F||P_AEP_{\widehat{A}^H}^\perp||_2) \\
&\qquad + \mathcal{O}(||E||_F||E||_2), \\
\frac{||\widehat{A}_p^-\widehat{A} - A^-A||_F}{\|A^-A||_F} &\leqslant ||A^\dagger||_2(||P_{\widehat{A}}EP_{A^H}^\perp||_2 + ||P_AEP_{\widehat{A}^H}^\perp||_2) + \mathcal{O}(||E||_F||E||_2).
\end{aligned} \tag{4.9}$$

By applying the results in Theorems 3.6 and 3.7, we have the following perturbation bounds for the nearest perturbed oblique projections under the spectral norm.

**Theorem 4.4.** *Suppose that $A, \widehat{A} = A + E \in \mathbf{C}^{m\times n}$ with $\operatorname{rank}(A) = \operatorname{rank}(\widehat{A}) = r > 0$ and $\|A^\dagger\|_2\|E\|_2 < 1$. For any given $A^- \in A\{1\}$, there exists a matrix $\widehat{A}_p^- \in \widehat{A}\{1\}$ such that*

$$\begin{aligned}
&(a)\ ||\widehat{A}\widehat{A}_p^- - AA^-||_2 \leqslant \max\left\{||\widehat{A}^\dagger||_2||I_m - AA^-||_2||EP_{\widehat{A}^H}||_2,\ \|A^\dagger\|_2||AA^-||_2||P_{\widehat{A}}^\perp EP_{A^H}||_2\right\}, \\
&\quad \frac{||\widehat{A}\widehat{A}_p^- - AA^-||_2}{\|AA^-||_2} \leqslant \max\left\{||\widehat{A}^\dagger||_2\frac{||I_m - AA^-||_2}{\|AA^-||_2}||EP_{\widehat{A}^H}||_2,\ \|A^\dagger\|_2||P_{\widehat{A}}^\perp EP_{A^H}||_2\right\}; \\
&(b)\ ||\widehat{A}_p^-\widehat{A} - A^-A||_2 \leqslant \max\left\{||\widehat{A}^\dagger||_2||I_n - A^-A||_2||P_{\widehat{A}}E||_2,\ \|A^\dagger\|_2||A^-A||_2||P_AEP_{\widehat{A}^H}^\perp||_2\right\}, \\
&\quad \frac{||\widehat{A}_p^-\widehat{A} - A^-A||_2}{\|A^-A||_2} \leqslant \max\left\{||\widehat{A}^\dagger||_2\frac{||I_n - A^-A||_2}{\|A^-A||_2}||P_{\widehat{A}}E||_2,\ \|A^\dagger\|_2||P_AEP_{\widehat{A}^H}^\perp||_2\right\}.
\end{aligned}$$

**Proof.** We only prove part (a). That of part (b) is similar. For any given $A^- \in A\{1\}$, let $\widehat{A}_p^-$ be as in Theorem 3.6. Then we observe that

$$\begin{aligned}
||(I_m - AA^-)\widehat{A}\widehat{A}^\dagger||_2 &= ||(I_m - AA^-)E\widehat{A}^\dagger||_2 \leqslant ||\widehat{A}^\dagger||_2||I_m - AA^-||_2||EP_{\widehat{A}^H}||_2, \\
||(I_m - \widehat{A}\widehat{A}^\dagger)AA^-||_2 &= ||P_{\widehat{A}}^\perp EP_{A^H}A^\dagger AA^-||_2 \leqslant \|A^\dagger\|_2||AA^-||_2||P_{\widehat{A}}^\perp EP_{A^H}||_2,
\end{aligned}$$

from which the inequalities in part (a) follow. □

## 5. Perturbation bounds for a consistent linear system

Wei in [19,20] studied rank deficient least squares problems. In this section, we discuss perturbation analysis of a system of consistent linear equations using the nearest perturbed g-inverses.

**Theorem 5.1.** *Suppose that* $A, \widehat{A} = A + E \in \mathbf{C}^{m\times n}$ *with* $\operatorname{rank}(\widehat{A}) = \operatorname{rank}(A) = r > 0$ *and* $\|A^\dagger\|_2\|E\|_2 < 1$, $b, \widehat{b} = b + \delta b \in \mathbf{C}^{m\times 1}$. *Consider the following system of linear equations*

$$Ax = b \tag{5.1}$$

*and its perturbed system*

$$\widehat{A}\widehat{x} = \widehat{b}, \tag{5.2}$$

*in which* (5.1) *is consistent. For any solution* $x = A^- b \neq 0$ *of the linear system* (5.1), *where* $A^-$ *has the form*

$$A^- = A^\dagger + A^\dagger AZ(I_m - AA^\dagger) + (I_n - A^\dagger A)Z, \quad Z \in \mathbf{C}^{n\times m}, \tag{5.3}$$

*there exists a solution* $\widehat{x} = \widehat{A}_m^- \widehat{b}$ *of the perturbed linear system* (5.2), *such that*

$$\begin{aligned}
\|\hat{x} - x\|_2 &\leqslant \|A^-\|_2\|\delta b\|_2 + ||A^\dagger||_2||P_{\widehat{A}}E||_2||x||_2 \\
&\quad + ||A^\dagger AZ(I_m - AA^\dagger)||_2||P_A^\perp E P_{\widehat{A}^H}\|_2\|x\|_2 + \mathcal{O}(\|E\|_2\|\delta b\|_2 + \|E\|_2^2),\\
\frac{\|\hat{x} - x\|_2}{\|x\|_2} &\leqslant \kappa(A)\left(\frac{\|\delta b\|_2}{\|b\|_2} + \frac{||P_{\widehat{A}}E||_2}{||A||_2} + \frac{||P_A^\perp E P_{\widehat{A}^H}\|_2}{||A||_2}\right) + \mathcal{O}(\|E\|_2\|\delta b\|_2 + \|E\|_2^2),
\end{aligned} \tag{5.4}$$

*in which* $\widehat{A}_m^-$ *has the form in* (3.1), *and* $\kappa(A) = \|A\|_2\|A^-\|_2$.

**Proof.** From Theorem 3.1, $\widehat{A}_m^- - A^- = \widehat{A}^\dagger - \widehat{A}^\dagger\widehat{A}A^-\widehat{A}\widehat{A}^\dagger$, therefore

$$\begin{aligned}
&\|(\widehat{A}_m^- - A^-)b\|_2 \\
&\quad = \|\widehat{A}^\dagger\widehat{A}[\widehat{A}^\dagger - A^\dagger - A^\dagger AZ(I_m - AA^\dagger) - (I_n - A^\dagger A)Z]\widehat{A}\widehat{A}^\dagger b\|_2 \\
&\quad = \|\{\widehat{A}^\dagger\widehat{A}[-\widehat{A}^\dagger EA^\dagger + \widehat{A}^\dagger(I_m - AA^\dagger) - A^\dagger AZ(I_m - AA^\dagger) - (I_n - A^\dagger A)Z]\widehat{A}\widehat{A}^\dagger\}b\|_2 \\
&\quad \leqslant ||\widehat{A}^\dagger||_2||P_{\widehat{A}}E||_2||[A^\dagger + (I_n - A^\dagger A)Z]\widehat{A}\widehat{A}^\dagger b||_2 \\
&\qquad + ||A^\dagger AZ(I_m - AA^\dagger)||_2||P_A^\perp E P_{\widehat{A}^H}\|_2\|\widehat{A}^\dagger b\|_2 + ||\widehat{A}^\dagger||_2^3||P_A^\perp E P_{\widehat{A}^H}||_2^2\|b\|_2.
\end{aligned}$$

Now (5.1) is consistent, so $b = Ax$ and

$$\begin{aligned}
&||[A^\dagger + (I_n - A^\dagger A)Z]\widehat{A}\widehat{A}^\dagger b||_2 \\
&\quad = ||[A^\dagger + (I_n - A^\dagger A)Z]\widehat{A}\widehat{A}^\dagger Ax||_2 \\
&\quad \leqslant ||[A^\dagger + (I_n - A^\dagger A)Z]Ax||_2 + ||[A^\dagger + (I_n - A^\dagger A)Z](I - \widehat{A}\widehat{A}^\dagger)Ax||_2 \\
&\quad \leqslant ||x||_2 + ||A^\dagger + (I_n - A^\dagger A)Z\|_2\|P_{\widehat{A}}^\perp E\|_2\|x||_2,\\
&||\widehat{A}^\dagger b||_2 = ||\widehat{A}^\dagger Ax||_2 \leqslant \|A^\dagger Ax||_2 + \|(\widehat{A}^\dagger - A^\dagger)Ax||_2 \\
&\qquad \leqslant \|x||_2 + 2\|\widehat{A}^\dagger\|_2\|E\|_2\|x||_2,
\end{aligned}$$

here we have used the fact that $[A^\dagger + (I_n - A^\dagger A)Z]Ax = A^-Ax = A^-b = x$. Therefore,

$$\begin{aligned}
\|(\widehat{A}_m^- - A^-)b\|_2 &\leqslant ||A^\dagger||_2||P_{\widehat{A}}E||_2||x||_2 + ||A^\dagger AZ(I_m - AA^\dagger)||_2||P_A^\perp E P_{\widehat{A}^H}\|_2\|x||_2 \\
&\quad + \mathcal{O}(\|E\|_2^2),\\
\|\hat{x} - x\|_2 &= \|\widehat{A}_m^-\widehat{b} - A^-b\|_2 \leqslant \|\widehat{A}_m^-\delta b\|_2 + \|(\widehat{A}_m^- - A^-)b\|_2 \\
&\leqslant \|A^-\|_2\|\delta b\|_2 + ||A^\dagger||_2||P_{\widehat{A}}E||_2||x||_2 + ||A^\dagger AZ(I_m - AA^\dagger)||_2||P_A^\perp E P_{\widehat{A}^H}\|_2\|x||_2 \\
&\quad + \mathcal{O}(\|E\|_2\|\delta b\|_2 + \|E\|_2^2),
\end{aligned}$$

$$\frac{\|\hat{x}-x\|_2}{\|x\|_2} \leqslant \|A^-\|_2\|A\|_2\frac{\|\delta b\|_2}{\|b\|_2} + ||A^\dagger||_2\|A\|_2\frac{||P_{\widehat{A}}E||_2}{||A||_2}$$
$$+ ||A^\dagger AZ(I_m - AA^\dagger)||_2\|A\|_2\frac{||P_A^\perp EP_{\widehat{A}^H}\|_2}{||A||_2} + \mathcal{O}(\|E\|_2\|\delta b\|_2 + \|E\|_2^2)$$
$$\leqslant \kappa(A)\left(\frac{\|\delta b\|_2}{\|b\|_2} + \frac{||P_{\widehat{A}}E||_2}{||A||_2} + \frac{||P_A^\perp EP_{\widehat{A}^H}\|_2}{||A||_2}\right) + \mathcal{O}(\|E\|_2\|\delta b\|_2 + \|E\|_2^2),$$

which completes the proof of the theorem. □

**Remark 5.1.** If we choose $A^- = A^\dagger$, then $Z = 0$ and $\kappa(A) = \|A\|_2\|A^\dagger\|_2$, the estimates in Theorem 5.1 is the same as for the consistent least squares problem discussed in [19]. Also notice that, when $r = m$, then $||P_A^\perp EP_{\widehat{A}^H}\|_2 = 0$.

**Remark 5.2.** In general, when (5.1) is consistent, we can not guarantee (5.2) is also consistent, therefore $\hat{x} = \widehat{A}_m^-\hat{b}$ may not be a solution of (5.2). However, when $A$ has full row rank and $\|A^\dagger\|_2\|E\|_2 < 1$, then $\widehat{A}$ also has full row rank, and in this case, $\hat{x} = \widehat{A}_m^-\hat{b}$ is indeed a solution of (5.2). Moreover, if we compute a solution of (5.1) using a stable algorithm, and if $\kappa(A)$ is of moderate size, then from Theorem 5.1 we see that, computed solution $\hat{x}$ has the formula $\hat{x} = \widehat{A}_m^-\hat{b}$, which is close to a true solution $x$ of (5.1).

## 6. Numerical experiments

In this section, we provide numerical experiments to verify the analysis of previous sections. The experiments are performed by MATLAB 7.3.0 on PC machine (Intel Celeron 2.66 GHz, Memory 512 MB), all functions are defined by MATLAB 7.3.0.

**Example 6.1.** Let

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} \epsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon & 0 & \epsilon \\ 0 & 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $\epsilon = 10^{-10}$, and $\widehat{A} = A + E$. Then we have $\operatorname{rank}(\widehat{A}) = \operatorname{rank}(A) = 3$, $\|A^\dagger\|_2\|E\|_2 \sim 1.0000\text{e}{-010} \ll 1$. Choose

$$A^- = \begin{pmatrix} 50.2500 & 0.2500 & 0.5000 & 0.5000 & 0 \\ 0 & -1.0000 & 0 & 0 & 1.0000 \\ 0.5000 & -0.5000 & 0.5000 & -0.5000 & 0 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 & 0 \\ -49.7500 & 0.2500 & -0.5000 & -0.5000 & 0.0100 \end{pmatrix}.$$

In Table 1, we report the numerical results derived in Theorems 4.1–4.4, and compare them with the exact distances of the nearest perturbed g-inverses and oblique projections. We observe that the derived perturbation bounds of the nearest perturbed g-inverses and oblique projections have the same order in size as the exact values themselves, which illustrate the perturbation bounds obtained in previous sections are sharp. The relative perturbation bounds are listed in Table 2.

**Example 6.2.** Let the matrices $A$, $A^-$, $E$ and $\widehat{A}$ be the same as in Example 6.1, further let $b = (0, 0, 1, 1, 0)^T$ and $\widehat{b} = b + \delta b$, where $\delta b = (0, \epsilon, \epsilon, 0, 0)^T$, $\epsilon = 10^{-10}$, then the linear system (5.1) is consistent.

**Table 1**
Absolute perturbation bounds for the nearest perturbed g-inverses and oblique projections.

| $\|\cdot\|$ | Frobenius norm | | Spectral norm | |
|---|---|---|---|---|
| | Exact values | Upper bounds | Exact values | Upper bounds |
| $\|\widehat{A}_m^- - A^-\|$ | 4.9763e−009 | 7.1241e−009 | 4.9763e−009 | 7.1220e−009 |
| $\|\widehat{A}\widehat{A}_p^- - AA^-\|$ | 1.0000e−010 | 1.8252e−010 | 8.6603e−011 | 1.6002e−010 |
| $\|\widehat{A}_p^-\widehat{A} - A^-A\|$ | 1.0001e−008 | 1.7324e−008 | 8.6611e−009 | 1.2249e−008 |

**Table 2**
Relative perturbation bounds for the nearest perturbed g-inverses and oblique projections.

| $\|\cdot\|$ | Frobenius norm | | Spectral norm | |
|---|---|---|---|---|
| | Exact values | Upper bounds | Exact values | Upper bounds |
| $\frac{\|\widehat{A}_m^- - A^-\|}{\|A^-\|}$ | 7.0339e−011 | 1.5000e−010 | 7.0366e−011 | 1.5000e−010 |
| $\frac{\|\widehat{A}\widehat{A}_p^- - AA^-\|}{\|AA^-\|}$ | 4.4720e−011 | 1.0000e−010 | 5.0000e−011 | 9.2388e−011 |
| $\frac{\|\widehat{A}_p^-\widehat{A} - A^-A\|}{\|A^-A\|}$ | 8.1636e−011 | 1.4142e−010 | 7.0710e−011 | 1.0000e−010 |

**Table 3**
Perturbation bounds for consistent linear system (5.1).

| $\|\cdot\|$ | Exact values | Upper bounds |
|---|---|---|
| $\|\widehat{x} - x\|_2$ | 1.9685e−010 | 1.0297e−008 |
| $\frac{\|\widehat{x}-x\|_2}{\|x\|_2}$ | 1.1365e−010 | 2.9146e−008 |

For solutions $x = A^-b$ of the consistent linear system (5.1) and $\widehat{x} = \widehat{A}_m^-\widehat{b}$ of the perturbed linear system (5.2), where $\widehat{A}_m^-$ has the form in (3.1), in Table 3 we list the numerical results derived in Theorem 5.1. We observe that the derived perturbation bounds in (5.4) for consistent linear system (5.1) are also sharp.

The above two examples and many other experiments we have carried out indicate that our analysis in this paper are valid.

## 7. Concluding remarks

For any given $A^- \in A\{1\}$, in this paper, we have derived the g-inverses $\widehat{A}^- \in \widehat{A}\{1\}$ such that the distances between the two g-inverses or oblique projections are the minimum under matrix Frobenius norm or spectral norm, and the corresponding distances have been established. Moreover, perturbation bounds for the nearest g-inverses, oblique projections of a matrix, and the consistent linear system have been derived, numerical experiments illustrate the validity of our analysis.

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